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The 2-Adic Affine Building of Type $\tilde{A}_2$ and Its Finite Projections

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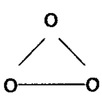
A certain "free" group U is constructed that is generated by three elements of order 3 which pairwise generate a Frobenius group of order 21 and it is shown that U operates regularly on the affine building of type $\tilde{A}_2$ over the field of 2-adic numbers. As a result an infinite series of finite rank 3 geometries is obtained whose rank 2 residues are projective planes of order 2, and which possess a regular automorphism group isomorphic to $SL_3(p)$ or $SU_3(p)$ for some prime p. © 1985 Academic Press, Inc.

This paper is a continuation of [5], where triangle groups were studied, which are groups generated by three elements of order 3 such that any two of them generate a Frobenius group of order 21. These are exactly the groups operating regularly on a rank 3 chamber system (Tits geometry) whose rank 2 residues are projective planes of order 2. It turned out that there are four different types of triangle groups—depending on the possible relations between each two of the generators of order 3—and we gave examples of finite triangle groups for three of these types. Most interesting was type 1 with examples $L_3(2)$, $U_3(3)$ and $U_3(5)$. The chamber systems of these groups possess a common universal 2-covering, which is constructed in this note as well as the free triangle group U with relations of type 1. A result of Tits [9, Corollary 3] says that the universal 2-covering $\mathscr{C}$ of a chamber system with diagram



is a building. In fact for the chamber system of a triangle group of type 1, $\mathscr{C}$ is isomorphic to the affine building of type $\tilde{A}_2$ over the 2-adic numbers $\mathbb{Q}_2$. The group U is shown to be isomorphic to $SU_3(R, f)$—for suitably chosen subring R of $\mathbb{Q}_2$ and unitary form f over R—which acts transitively on the affine building of $SL_3(\mathbb{Q}_2)$. Moreover, we get finite projections isomorphic to $SL_3(p)$ or $SU_3(p)$, depending on the odd prime p, which provide an

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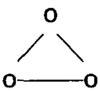
infinite series of finite simple triangle groups (after passage modulo the center).

Of course, these examples also give infinitely many non-isomorphic geometries of diagram



with flag transitive automorphism group. We recall that the triangle group $G=\langle a, b, c\rangle$ with generators a, b, c of order 3 is of type 1, iff the relations $(ab)^2=ba$, $(bc)^2=cb$ and $(ca)^2=ac$ hold [5, 7].

Parts of this work were inspired by Kantor's papers [3, 4] and a colloquium talk given by Tits.

Let $\mathbb{Q}_2$ be the field of 2-adic numbers and $\mathbb{Z}_2$ its ring of integers and R, S be the subrings of $\mathbb{Q}_2$ generated by 2^{-1}, x and 7^{-1}, x, respectively, where x is a root of the equation $z^2-z+2=0$. That such an element x exists in $\mathbb{Z}_2$ follows directly from the fact that -7 is a square in $\mathbb{Z}_2$. Moreover, since x and $1-x$ are the two roots of the equation $z^2-z+2=0$ we may even assume that x is a unit in $\mathbb{Z}_2$.

Let α, β be the following elements of $SL_3(\mathbb{Z}_2)$:

$$\alpha=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ x-1 & -1 & -1 \end{pmatrix}, \qquad \beta=\begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -x \\ 0 & 0 & 1 \end{pmatrix}.$$

Then it is easily checked that $o(\alpha)=o(\beta)=3$, $\alpha\neq\beta^{-1}$ and $(\alpha\beta)^2=\beta\alpha$, which shows that $\langle\alpha, \beta\rangle$ is a Frobenius group of order 21. In fact this choice of α, β is almost canonical; since $\langle\alpha, \beta\rangle$ operates irreducibly on $(\mathbb{Q}_2)^3$ and as $\mathbb{Q}_2$ possesses no seventh root of unity an element σ of order 7 in $\langle\alpha, \beta\rangle$ has a minimal polynomial of degree 3, either $t^3-(x-1)\,t^2-xt-1$ or $t^3+xt^2+(x-1)\,t-1$. Choosing $\sigma=\alpha\beta$ as the companion matrix of the first one and α such that it fixes the first basis vector gives α, β as above.

If now $\rho\in GL_3(\mathbb{Q}_2)$ is such that $\alpha^\rho=\beta$ and ρ^3 centralizes α then with $\gamma=\beta^\rho$, $\langle\beta, \gamma\rangle$ and $\langle\gamma, \alpha\rangle$ are also Frobenius groups of order 21. That means that $\langle\alpha, \beta, \gamma\rangle$ is a triangle group of type 1 [5, Sect. 1] and ρ acts as a graph automorphism of order 3.

A straightforward calculation shows that up to multiplication with scalars there are only two choices for ρ:

$$\rho_1=\begin{pmatrix} 0 & 0 & 2^{-1}(x-2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \qquad \rho_2=(2x-1)^{-1}\cdot\begin{pmatrix} 0 & 0 & 2x-1 \\ 3 & 1 & x-1 \\ -1 & 2 & -1 \end{pmatrix}.$$

We see that $\rho_1 \in GL_3(R)$, $\rho_2 \in SL_3(S)$ due to $(2x-1)^2 = -7$.

Further there is a unique involutory automorphism $y \to \bar{y}$ of R and likewise of S interchanging x and $1-x$; this can be most easily seen by considering both rings as subrings of $\mathbb{C}$ with complex conjugation. Now there is a unique matrix ω, hermitian with respect to $\bar{\ }$, such that both α and β satisfy

$$\alpha\omega\bar{\alpha}' = \omega; \qquad \beta\omega\bar{\beta}' = \omega,$$

namely,

$$\omega = \begin{pmatrix} -3 & x & x \\ \bar{x} & -3 & x \\ \bar{x} & \bar{x} & -3 \end{pmatrix}.$$

And as $\det \omega = -7$, ω gives rise to a non-degenerate unitary form f over R (and likewise over S) such that α, β respect this form. It turns out that ρ_1 and ρ_2 also respect f, so with

$$\gamma = \beta^{\rho_1} = \begin{pmatrix} -1 & -x & 2^{-1}(2-x) \\ 0 & 1 & 0 \\ -2^{-1}(x+1) & 0 & 0 \end{pmatrix},$$

$$\delta = \beta^{\rho_2} = 7^{-1} \cdot \begin{pmatrix} 2x-8 & -4x-5 & -5x+6 \\ -2x-6 & -3x+5 & -2x+1 \\ 3-6x & 1-2x & x+3 \end{pmatrix}$$

and $U = \langle \alpha, \beta, \gamma \rangle$ we have $U \subseteq SU_3(R, f)$ and $\langle \alpha, \beta, \delta \rangle \subseteq SU_3(S, f)$. We will later see that $U = SU_3(R, f)$ and $\langle \alpha, \beta, \delta \rangle \simeq L_3(2)$.

Even without this we can reduce modulo prime ideals of R and exhibit an infinite sequence of triangle groups. In fact if p is an odd prime such that -7 is not a square in $GF(p)$—this happens exactly for $p \equiv 3, 5, 6 \bmod 7$—then there is a ring homomorphism from R onto $GF(p^2)$ such that the automorphism $\bar{\ }$ becomes the involutory automorphism of $GF(p^2)$. Hence it induces a group homomorphism φ_p from $SU_3(R, f)$ into $SU_3(p)$. If p is an odd prime such that -7 is a square then we have a ring homomorphism from R onto $GF(p)$ inducing a group homomorphism φ_p from $SU_3(R, f)$ into $SL_3(p)$. With that we have:

THEOREM 1. *The following groups are triangle groups of type* 1:

(a) $SU_3(p)$ *for p prime*, $p \equiv 3, 5, 6 \bmod 7$;

(b) $SL_2(7)$;

(c) $SL_3(p)$ *for p prime*, $p \equiv 1, 2, 4 \bmod 7$.

Proof. Since for $p=2$ $SL_3(2)$ is already known to be a triangle group [5, 6], we may assume $p \neq 2$. Then it suffices to show that—except for $p=7$—$\varphi_p(U)$ is always the full group $SU_3(p)$ or $SL_3(p)$. An inspection of the list of all subgroups of these groups—see, e.g., [1, 6]—shows that apart from $SL_3(p)$ and $SU_3(p)$ the perfect subgroups of $SL_3(p)$ containing a Frobenius group of order 21 are isomorphic to A_7 or $L_3(2)$. As A_7 is not a triangle group of type 1, and as $\varphi_p(\alpha\beta^2\gamma)$ is never an involution, the result is a consequence of Lemma 2 below. For $p=7$ we see that $^-$ is projected onto the identity of $GF(7)$ and hence $\varphi_7(U)$ is orthogonal with respect to the (singular) matrix

$$\begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}$$

having a 2-dimensional radical. Thus $\varphi_7(U)$ is contained in some maximal parabolic subgroup of $SL_3(7)$. But since $\varphi_7((\alpha\beta^2\gamma)^2)$ is not central in $\varphi_7(U)$, we see that $\varphi_7(U)$ is the whole maximal parabolic subgroup isomorphic to $E_{49}SL_2(7)$. Now (b) is obvious. By Lemma 2 one can also verify that $\langle \alpha, \beta, \delta \rangle \subseteq SU_3(S, f)$ is isomorphic to $L_3(2)$. Now reduction modulo p (for $p \neq 7$) gives rise to the somewhat exceptional $L_3(2)$ in the subgroup list of $SL_3(p)$ and $SU_3(p)$.

LEMMA 2. *Let G be a triangle group, $G = \langle a, b, c \rangle$, where a, b, c satisfy the relations* (1). *Then $G \simeq L_3(2)$ if and only if $o(ab^2c) = 2$.*

Proof. If $o(ab^2c) = 2$ then coset enumeration gives 8 cosets of $\langle a, b \rangle$ in G. Now $L_3(2)$ is the only perfect group of order 168 and hence by [5, Proposition 1], $G \simeq L_3(2)$. If on the other hand $G \simeq L_3(2)$ then one can easily verify that given a and b, the element c satisfying the relations (1) is unique. Thus $o(ab^2c) = 2$ can be checked using [5, Example 2].

Considering the situation more geometrically we associate with each triangle group G its chamber system $\mathscr{C}(G; 1; \langle a \rangle, \langle b \rangle, \langle c \rangle)$ where a, b, c are the three canonical generators of order 3. This chamber system consists of the set of all elements of G equipped with the partitions P_a, P_b, P_c given by: $x, y \in G$ lie in the same element of P_s, if $x\langle s \rangle = y\langle s \rangle$, $s \in \{a, b, c\}$. In this terminology the group homomorhism φ_p induces a 2-covering of the corresponding chamber systems. We will show in Theorem 7 that the chamber system $\mathscr{C}(U; 1; \langle \alpha \rangle, \langle \beta \rangle, \langle \gamma \rangle)$ is isomorphic to the affine building of type $\tilde{A}_2$ over $\mathbb{Q}_2$ and hence the 2-coverings induced by φ_p of the finite examples in Theorem 1 are universal 2-coverings.

We briefly describe the affine building of type $\tilde{A}_2$ over $\mathbb{Q}_2$. It is defined using the group $SL_3(\mathbb{Q}_2)$ and is conveniently explained via certain $\mathbb{Z}_2$-lattice stabilizers:

Indeed let L_0, L_1, L_2 be the $\mathbb{Z}_2$-lattices

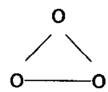
$$L_0 = \langle e_1, e_2, e_3 \rangle_{\mathbb{Z}_2},$$
$$L_1 = \langle 2^{-1}e_1, e_2, e_3 \rangle_{\mathbb{Z}_2},$$
$$L_2 = \langle e_1, e_2, 2e_3 \rangle_{\mathbb{Z}_2},$$

where e_1, e_2, e_3 form the standard basis of the 3-dimensional vector space over $\mathbb{Q}_2$. Moreover, let H_i be the stabilizer of L_i in $SL_3(\mathbb{Q}_2)$. Then clearly $H_0 = SL_3(\mathbb{Z}_2)$ and H_1, H_2 consist of matrices of the form

$$H_1\colon \begin{pmatrix} a & 2^{-1}b & 2^{-1}c \\ 2d & e & f \\ 2g & h & i \end{pmatrix} \qquad H_2\colon \begin{pmatrix} a & b & 2^{-1}c \\ d & e & 2^{-1}f \\ 2g & 2h & i \end{pmatrix}$$

where $a, \ldots, i \in \mathbb{Z}_2$. Finally, let $B = H_0 \cap H_1 \cap H_2$.

THEOREM 3 ([2]). *The chamber system* $\mathscr{C}(SL_3(\mathbb{Q}_2); B; H_0 \cap H_1, H_1 \cap H_2, H_2 \cap H_0)$ *is a building with diagram*



and the rank 2 *residues are projective planes of order* 2.

The basic observation now is that $\langle \alpha, \beta \rangle \subseteq H_0$, $\langle \alpha, \gamma \rangle \subseteq H_1$, $\langle \beta, \gamma \rangle \subseteq H_2$. This rests on the fact that we could choose x as a unit of $\mathbb{Z}_2$: as a result $x-2$ is a unit too and so $-2^{-1}(x+1) = (-2^{-1}(x-2))^{-1} \in 2\mathbb{Z}_2$. Since the groups $\langle \alpha, \beta \rangle$, $\langle \alpha, \gamma \rangle$ and $\langle \beta, \gamma \rangle$ have trivial intersection with B, we get $H_i = (U \cap H_i)\, B$ for $i = 0, 1, 2$. This suggests the following:

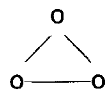
LEMMA 4. *U operates transitively on the chamber system* $\mathscr{C}(SL_3(\mathbb{Q}_2); B; H_0 \cap H_1, H_1 \cap H_2, H_2 \cap H_0)$.

Proof. We have to show that $SL_3(\mathbb{Q}_2) = U \cdot B$. From the remarks above we have $H_i = (U \cap H_i)\, B = B(U \cap H_i)$ for $i = 0, 1, 2$. Now $SL_3(\mathbb{Q}_2)$ is generated by H_0, H_1, H_2, i.e., the building is connected, thus any element $\kappa \in SL_3(\mathbb{Q}_2)$ is a product of elements of H_0, H_1, H_2. We prove that $\kappa \in U \cdot B$ by induction on the length of κ in a minimal product representation. By induction $\kappa = \mu\lambda$ for some $\lambda \in H_i$, $\mu \in UB$. But then $\lambda \in (U \cap H_i)\, B$ and hence $\kappa = \mu\lambda \in UB(U \cap H_i)\, B = U(U \cap H_i)\, B = UB$.

As a consequence the group $SU_3(R, f)$ also operates transitively on the chamber system. We even have:

LEMMA 5. *$SU_3(R,f)$ operates regularly on the chamber system $\mathscr{C}(SL_3(\mathbb{Q}_2); B; H_0 \cap H_1, H_1 \cap H_2, H_2 \cap H_0)$.*

Proof. For abbreviation we put $G = SU_3(R, f)$. It remains to show that $B \cap G = 1$. In fact there are only finitely many elements in $G \cap H_0$, since after some change of basis the rows of these elements are solutions of some diophantine equations with positive coefficients—thus $B \cap G \subseteq H_0 \cap G$ is finite (this argument is made after Lemma 3.2 in [3]). Now, for suitably large primes p, we have that the chamber system $\mathscr{C}(\varphi_p(G); \varphi_p(G \cap B); \varphi_p(G \cap H_0 \cap H_1), \varphi_p(G \cap H_1 \cap H_2), \varphi_p(G \cap H_0 \cap H_2))$ has diagram



By [8, Theorem 2], $\varphi_p(G \cap B) \subset Z(\varphi_p(G))$. Hence $G \cap B$ consists only of diagonal matrices; but as R has no non-trivial cubic roots of unity we conclude $G \cap B = 1$.

Combining these two lemmas we immediately get:

COROLLARY 6. $U = SU_3(R, f)$.

THEOREM 7. *The chamber system $\mathscr{C}(U; 1; \langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle)$ is isomorphic to the affine building of type $\tilde{A}_2$ over $\mathbb{Q}_2$.*

This has two interesting consequences:

THEOREM 8. *$U = SU_3(R, f)$ is freely generated by α, β, γ subject to the relations of type* 1.

Proof. Let G be the free group generated by a, b, c with relations of type 1. Then there is a unique homomorphism φ from G onto U with $\varphi(a) = \alpha$, $\varphi(b) = \beta$, $\varphi(c) = \gamma$. But since $\langle a, b\rangle$, $\langle b, c\rangle$, $\langle c, a\rangle$ are Frobenius groups of order 21, φ induces an isomorphism of the rank 2 residues of the corresponding chamber systems and hence is a 2-covering. As the chamber system associated with U is a building, it is simply 2-connected [9, Theorem 3], so φ must be an isomorphism.

The other consequence is the fact that the chamber system $\mathscr{C}(U; 1; \langle\alpha\rangle, \langle\beta\rangle, \langle\gamma\rangle)$ is the universal 2-cover of the chamber systems associated with the finite groups in Theorem 1. In fact a universal 2-cover of each of those chamber systems exists by [9, 5.11]; this, however, must therefore cover the chamber system corresponding to U and since that is simply 2-connected they have to be isomorphic.

Finally, we note that it follows from [5, Lemma 11] that all quotients of U, i.e., all triangle groups of type 1—with the single exception of $L_3(2)$—are flag-transitive in the sense of [5].

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